Towards a measurement of quantum backflow

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Quantum backflow is a classically forbidden effect consisting in a negative flux for states with negligible negative momentum components. It has never been observed experimentally so far. We derive a general relation that connects backflow with a critical value of the particle density, paving the way for the detection of backflow by a density measurement. To this end, we propose an explicit scheme with Bose-Einstein condensates, at reach with current experimental technologies. Remarkably, the application of a positive momentum kick, via a Bragg pulse, to a condensate with a positive velocity may cause a current flow in the negative direction.

Quantum backflow is a fascinating quantum interference effect consisting in a negative current density for quantum wave packets without negative momentum components [1]. Despite its intriguing nature - obviously counterintuitive from a classical viewpoint - quantum backflow has not yet received as much attention as other quantum effects. Firstly discovered by Allcock in 1969 [1], it only started to attract some attention in the mid 90's. Bracken and Melloy [2] provided a bound for the maximal fraction of probability that can undergo backflow. Then, additional bounds and analytic examples where discussed by Muga *et al.* [3–5]. Recently, Berry [6] analyzed the statistics of backflow for random wavefunctions, and Yearsley *et al.* [7] studied some specific cases, clarifying the maximal backflow limit. However, so far no experiments have been performed, and a clear program to carry out one is also missing. Two main difficulties are the measurement of the current density (the existing proposals for local and direct measurements are rather idealized schemes [8]) and the preparation of the necessary states.

In this Letter we derive a general relation that connects the current and the particle density, allowing for the detection of backflow by a density measurement, and propose a scheme for its observation with Bose-Einstein condensates in harmonic traps, that could be easily implemented with current experimental technologies. In particular, we show that preparing a Bose-Einstein condensate with positive-momentum components, and then further transferring a positive momentum kick to part of the atoms, causes under certain conditions, remarkably, a current flow in the negative direction.

Let us start by considering a one dimensional Bose-Einstein condensate with a narrow momentum distribution centered around $\hbar k_1 > 0$, with negligible negative components. Then, we apply a Bragg pulse that transfers a momentum $\hbar q > 0$ to part of the atoms [9], populating a state of momentum $\hbar k_2 = \hbar k_1 + \hbar q$ (see Fig. 1). By indicating with A_1 and A_2 the amplitudes of the two
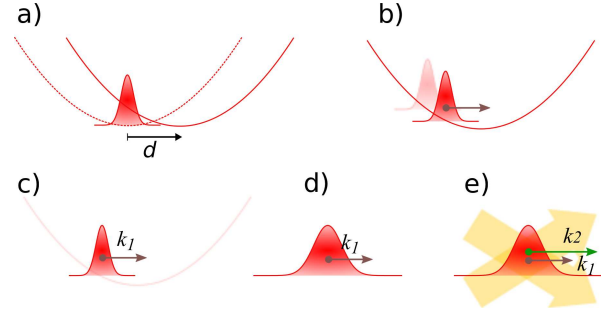

FIG. 1. (Color online) a) A condensate is created in the ground state of a harmonic trap with frequency ω_x; at $t = 0$ we apply a magnetic gradient that shifts the trap by a distance d; b) the condensate starts to perform dipole oscillations in the trap; c) when it reaches a desired momentum $\hbar k_1$ the trap is switched off, and d) then the condensate is let expand for a time t; e) finally, a Bragg pulse is applied in order to transfer part of the atoms to a state of momentum $\hbar k_2$.

momentum states, the total wave function is

$$\Psi(x,t) = \psi(x,t)\left(A_1 + A_2 \exp\left[iqx + i\varphi\right]\right), \quad (1)$$

where we can assume $A_1, A_2 \in \mathbb{R}^+$ without loss of generality (with $A_1^2 + A_2^2 = 1$), φ being an arbitrary phase. All these parameters (except the phase, that will be irrelevant in our scheme), can be controlled and measured in the experiment. Then, by writing the wave function of the initial wave packet as

$$\psi(x,t) = \phi(x,t)\exp[i\theta(x,t)], \quad (2)$$

the expression for the total current density, $J_\Psi(x,t) = (\hbar/m)\text{Im}\left[\Psi^*\nabla\Psi\right]$ can be easily put in the form

$$\frac{m}{\hbar}J_\Psi(x,t) = (\nabla\theta)\rho_\Psi + \frac{1}{2}q\left[\rho_\Psi + |\phi|^2(A_2^2 - A_1^2)\right], \quad (3)$$

with $\rho_\Psi(x,t) = |\phi(x,t)|^2\left(A_1^2 + A_2^2 + 2A_1A_2\cos(qx + \varphi)\right)$ being the total density. Therefore, a negative flux,

$J_\Psi(x,t) < 0$, corresponds to the following inequality for the density

$$\text{sign}[\eta(x,t)]\rho_\Psi(x,t) < \frac{1}{|\eta(x,t)|}|\phi(x,t)|^2(A_1^2 - A_2^2), \quad (4)$$

where we have defined $\eta(x,t) = 1 + 2\nabla\theta(x,t)/q$. Later on we will show that $\eta(x,t) < 0$ corresponds to a *classical regime*, whereas for $\eta(x,t) > 0$ the backflow is a purely quantum effect, without any classical counterpart. Therefore, in the *quantum regime*, backflow takes place when the density is below the following critical threshold:

$$\rho_\Psi^{crit}(x,t) = \frac{q}{q + 2\nabla\theta(x,t)}|\phi(x,t)|^2(A_1^2 - A_2^2). \quad (5)$$

This is a fundamental relation that allows to detect backflow by a density measurement. It applies to any class of wavepackets of the form (1), including the superposition of two plane waves discussed in [5, 7].

In order to propose a specific experimental implementation, we consider a condensate in a three dimensional harmonic trap, with axial frequency ω_x. We assume a tight radial confinement, $\omega_\perp \gg \omega_x$, so that the wave function can be factorized in a radial and axial components (in the non interacting case this factorization is exact)[10]. In the following we will focus on the 1D axial dynamics, taking place in the waveguide provided by the transverse confinement, that is assumed to be always on. We define $a_x = \sqrt{\hbar/m\omega_x}$, $a_\perp = \sqrt{\hbar/m\omega_\perp}$.

The scheme proceeds as highlighted in Fig.1. The condensate is initially prepared in the ground state ψ_0 of the harmonic trap. Then, at $t = 0$ the trap is suddenly shifted spatially by d (Fig.1a) and the condensate starts to perform dipole oscillations (Fig.1b). The corresponding wave function is given by a scaling transformation [11]

$$\psi(x,t) = \psi_0(x - \xi(t))\exp[ik(t)x - i\gamma(t)], \quad (6)$$

with $\xi(t) = d[1 - \cos(\omega_x t)]$, $k(t) = (m/\hbar)\omega_x d\sin(\omega_x t)$, the value of the phase $\gamma(t)$ being irrelevant. Next, at $t = t_1$, when the condensate has reached a desired momentum $mv_1 = \hbar k_1 = \hbar k(t_1)$, the trap is switched off (Fig.1c). For convenience we redefine the spatial coordinate as $x - \xi(t_1) \to x$, and we set $\gamma_1 = \gamma(t_1)$. Therefore, the wave function reads

$$\psi(x,t_1) = \psi_0(x)\exp(ik_1 x - i\gamma_1). \quad (7)$$

At this point we let the condensate expand freely for a time t (Fig.1d). Hereinafter we will consider explicitly two cases, namely a noninteracting condensate and the Thomas-Fermi (TF) limit [11], that can both be treated analytically. In fact, in both cases the expansion can be expressed by a scaling transformation

$$\psi(x,t) = \frac{1}{\sqrt{b(t)}}\psi_0\left(\frac{x - v_1 t}{b(t)}\right) \quad (8)$$

$$\times \exp\left[i\frac{m}{2\hbar}x^2\frac{\dot{b}(t)}{b(t)} + ik_1 x\left(1 - \frac{\dot{b}}{b}t\right) + i\beta(t)\right],$$

where $b(t)$ represents the scaling parameter (for convenience we have also redefined the time coordinate, $t - t_1 \to t$) and $\beta(t)$ is an irrelevant global phase. This expression can be easily obtained by generalizing the scaling in [12, 13] to the case of an initial velocity field.

In the *non interacting case*, the initial wave function is a minimum uncertainty Gaussian, $\psi_0(x) = (1/\pi^{\frac{1}{4}}\sqrt{a_x})\exp[-x^2/(2a_x^2)]$, and the scaling parameter evolves as $b(t) = \sqrt{1 + \omega_x^2 t^2}$ [11, 14]. For a TF distribution we have $\psi_0(x) = [(\mu - \frac{1}{2}m\omega_x^2 x^2)/g_{1D}]^{1/2}$ for $|x| < R_{TF} \equiv \sqrt{2\mu/m\omega_x^2}$ and vanishing elsewhere, with $g_{1D} = g_{3D}/(2\pi a_\perp^2)$ [15], and the chemical potential μ fixed by the normalization condition $\int dx|\psi|^2 = N$, the latter being the number of atoms in the condensate [11]. In this case $b(t)$ satisfies $\ddot{b}(t) = \omega_x^2/b^2(t)$, whose asymptotic solution, for $t \gg 1/\omega_x$, is $b(t) \simeq \sqrt{2}t\omega_x$ [16].

Finally, we apply a Bragg pulse as discussed previously (Fig.1e). We may safely assume the duration of the pulse to be very short with respect to the other timescales of the problem [17]. Then, the resulting wave function is that in Eq. (1), with the corresponding critical density for backflow in Eq. (5). We have

$$\phi(x) = \frac{1}{\sqrt{b(t)}}\psi_0\left(\frac{x - (\hbar k_1/m)t}{b(t)}\right), \quad (9)$$

while the expression for the phase gradient is

$$\nabla\theta = \frac{m}{\hbar}x\frac{\dot{b}(t)}{b(t)} + k_1\left(1 - \frac{\dot{b}(t)}{b(t)}t\right) \quad (10)$$

that, in the asymptotic limit $t \gg 1/\omega_x$, yields the same result $\nabla\theta = mx/\hbar t$ for both the non interacting and TF wave packets. Eventually, backflow can be probed by taking a snapshot of the interference pattern just after the Bragg pulse, measuring precisely its minimum, and comparing it to the critical density.

Before proceeding to the quantum backflow, let us discuss the occurrence of a classical backflow. To this end it is sufficient to consider the flux of a single wave packet (before the Bragg pulse), namely $J_\psi(x,t) = (\hbar/m)|\phi|^2\nabla\theta$. In this case the flux is negative for $x < v_1(t - b/\dot{b}) =: x_-(t)$ (see Eq. (10)). Then, by indicating with R_0 the initial half width of the wave packet and with $R_L(t) = -b(t)R_0 + v_1 t$ its left border at time t, a negative flux occurs when $R_L < x_-$, that is for $R_0 > v_1/\dot{b}$. In the asymptotic limit, the latter relation reads $R_0 > fv_1/\omega_x$ ($f = 1, 1/\sqrt{2}$ for the non interacting and TF cases, respectively). On the other hand, the momentum width of the wave packet is $\Delta_p \approx \hbar/R_0$ [19], so that the negative momentum components can be safely neglected only when $mv_1 \gg \hbar/R_0$. From these two conditions, we get that there is a negative flux (even in the absence of initial negative momenta) when $R_0 \gg a_x$. This can be easily satisfied in the TF regime. In fact, in that case the backflow has a classical counterpart due to
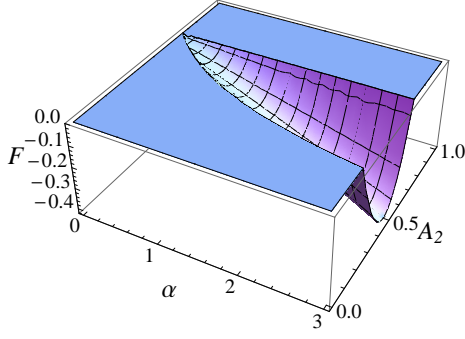
FIG. 2. (Color online) Plot of the function $F(\alpha, A_2)$ defined in Eq. (13). For a given value of α, the maximal backflow is obtained for the value A_2 that minimizes F.

FIG. 3. (Color online) Plot of the flux $J_\Psi(x)$. Backflow corresponds to $J_\Psi < 0$. Positions are measured with respect to the wave packet center.

the force $F = -\partial_x(g_{1D}|\rho_\psi(x,t)|^2)$ implied by the repulsive interparticle interactions [12]. These interactions are responsible for the appearance of negative momenta and backflow. Then, sufficient conditions for avoiding these classical effects are $k_1 \gg 1/a_x$ and $R_0 < fv_1/\omega_x$.

Let us now turn to the quantum backflow. In order to discuss the optimal setup for having backflow, it is convenient to consider the following expression for the current density,

$$J_\Psi(x,t) = \frac{\hbar}{m}|\phi|^2 \left[q\left(A_2^2 + A_1 A_2 \cos(qx + \varphi)\right) \quad (11) \right.$$
$$\left. + \nabla\theta \left(A_1^2 + A_2^2 + 2A_1 A_2 \cos(qx + \varphi)\right)\right],$$

that follows directly from Eqs. (3) and (5). Let us focus on its behavior around the center of wave packet, namely at $x \approx (\hbar k_1/m)t = d\omega_x t$ (the following analysis extends to the whole packet if $d\omega_x t$ is much larger than the condensate width). In the asymptotic limit, the phase gradient at the center is $\nabla\theta|_c \approx k_1$, and the flux in Eq. (11) turns out to be proportional to that of the superposition of two plane waves of momenta k_1 and $k_2 = k_1 + q$. This limit is particularly useful because for two plane waves the probability density is a sinusoidal function and the critical density becomes a constant, which in practice makes irrelevant the value of the arbitrary phase φ we cannot control. Then, the condition for having backflow at the wave packet center is

$$k_1 A_1^2 + k_2 A_2^2 + (k_1 + k_2)A_1 A_2 \cos(qx + \varphi) < 0. \quad (12)$$

Since all the parameters k_i and A_i are positive, the minimal condition for having a negative flux is $k_1 A_1^2 + k_2 A_2^2 < (k_1 + k_2)A_1 A_2$ (for $\cos(\cdot) = -1$), that can be written as

$$F(\alpha, A_2) \equiv 1 + \alpha A_2^2 - (2 + \alpha)A_2\sqrt{1 - A_2^2} < 0, \quad (13)$$

where we have defined $\alpha = q/k_1$. The behavior of the function $F(\alpha, A_2)$ in the region where $F < 0$ is depicted in Fig. 2. In particular, for a given value of the relative momentum kick α, the minimal value of F is obtained
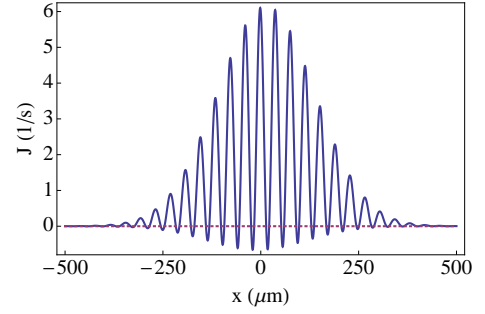
for A_2 that solves $\partial F/\partial A_2|_\alpha = 0$, that is for

$$2\alpha A_2\sqrt{1 - A_2^2} + (2 + \alpha)(2A_2^2 - 1) = 0. \quad (14)$$

In order to maximize the effect of backflow and its detection, one has to satisfy a number of constraints. In principle, Fig. 2 shows that the larger the value of $\alpha = q/k_1$, the larger the effect of backflow is. However, q cannot be arbitrarily large as it fixes the wavelength $\lambda = 2\pi/q$ of the density modulations, which must be above the current experimental spatial resolution σ_r, $\lambda \gg \sigma_r$, for allowing a clean experimental detection of backflow (see later on). In addition, as discussed before, k_1 should be sufficiently large for considering negligible the negative momentum components of the initial wave packet, $k_1 \gg 1/a_x$. Therefore, since the maximal momentum that the condensate may acquire after a shift d of the trap is $\hbar k_1 = m\omega_x d$, the latter condition reads $d \gg a_x$. By combining the two conditions above, we get the hierarchy $1 \ll d/a_x \ll (2\pi/\alpha)(a_x/\sigma_r)$. Furthermore, we recall that in the interacting case we must have $R_0 < fv_1/\omega_x = fd$ in order to avoid classical effects. Therefore, given the value of the current imaging resolution, the non interacting case ($R_0 \approx a_x$) appears more favorable than the TF one (where typically $R_{TF} \gg a_x$). Nevertheless, the latter condition can be substantially softened if the measurement is performed at the wave packet center, away from the left tail where classical effects take place.

As a specific example, here we consider the case of an almost noninteracting ^{7}Li condensate [20] prepared in the ground state of a trap with frequency $\omega_x = 2\pi \times 1$ Hz (yielding $a_x \simeq 38$ μm). Then, we shift the trap by $d = 80$ μm, so that after a time $t_1 = \pi/(2\omega_x) = 250$ ms the condensate has reached its maximal velocity $\hbar k_1/m = \omega_x d \simeq 0.5$ mm/s. At this point the axial trap is switched off, and the condensate is let expand for a time $t \gg a_x/\omega_x d$ until it enters the asymptotic plane wave regime (here we use $t = 1$ s). Finally we apply a Bragg pulse of momentum $\hbar q = \alpha\hbar k_1$, with $\alpha = 3$, that transfers 24% of the population to the state of momentum $\hbar k_2$, according to Eq. (14) ($A_2 = 0.49$, $A_1 \simeq 0.87$) [9]. The
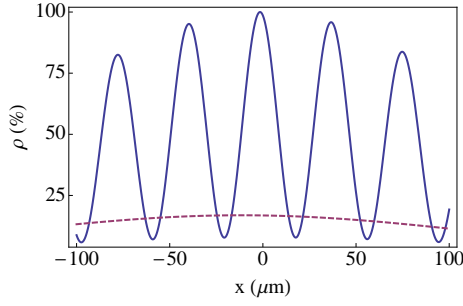
FIG. 4. (Color online) Density (solid) and critical density (dashed) for the case discussed in the text. Backflow occurs in the regions where the density is below the critical value, see Fig. 3. Positions are measured with respect to the wave packet center.

resulting flux is shown in Fig. 3, where the backflow is evident, and more pronounced at the wave packet center. The corresponding density is displayed in Fig. 4, where it is compared with the critical value of Eq. (5). The values obtained around the center are $\rho_\Psi^{min} \simeq 8\%$ and $\rho_\Psi^{crit} \simeq 17\%$.

Let us now discuss more thoroughly the implication of a finite imaging resolution σ_r. First we note that experimentally it is difficult to obtain a precise measurement of the absolute density, because of uncertainties in the calibration of the imaging setup. Instead, measurements in which the densities at two different points are compared are free from calibration errors and therefore are more precise. Owing to this, it is useful to normalize the total density $\rho_\Psi(x,t) = |\phi(x,t)|^2 \left(A_1^2 + A_2^2 + 2A_1A_2 \cos(qx+\varphi)\right)$ to its maximal value $\rho_\Psi^{max} \simeq |\phi^{max}|^2 \left(A_1^2 + A_2^2 + 2A_1A_2\right)$. In addition, we have to take into account that, due to the finite resolution σ_r [21], the sinusoidal term $\cos(qx+\varphi)$ is reduced by a factor $\zeta = \exp[-q^2\sigma_r^2/2]$ after the imaging. Then, by indicating with $x_{min}(t)$ the position of the density minima, we have

$$\frac{\rho_\Psi(x_{min},t)}{\rho_\Psi^{max}}\bigg|_{exp} = \frac{|\phi(x_{min},t)|^2}{|\phi^{max}|^2}\frac{A_1^2 + A_2^2 - 2\zeta A_1A_2}{A_1^2 + A_2^2 + 2\zeta A_1A_2},$$

where "exp" refers to the experimental conditions. Instead, the normalized critical density is (close the wave packet center, where $\nabla\theta \approx k_1$)

$$\frac{\rho_\Psi^{crit}(x,t)}{\rho_\Psi^{max}} = \frac{q}{q+2k_1}\frac{|\phi(x,t)|^2}{|\phi^{max}|^2}\frac{A_1^2 - A_2^2}{(A_1+A_2)^2}, \qquad (15)$$

so that, assuming that $|\phi(x,t)|^2$ varies to be on a scale much larger than σ_r to be unaffected by the finite imaging resolution, the condition for *observing* a density drop below the critical value reads

$$\frac{A_1^2 + A_2^2 - 2\zeta A_1A_2}{A_1^2 + A_2^2 + 2\zeta A_1A_2} = \frac{\alpha}{\alpha+2}\frac{A_1 - A_2}{A_1 + A_2}. \qquad (16)$$

In particular, in the example case we have discussed, backflow could be clearly detected with an imaging resolution of about $3\ \mu$m, which is within reach of current experimental setups.

In conclusion, we have presented a feasible experimental scheme that could lead to the first observation of quantum backflow, namely the presence of a negative flux for states with negligible negative momentum components. By using current technologies for ultracold atoms, we have discussed how to imprint backflow on a Bose-Einstein condensate and how to detect it by a usual density measurement. Remarkably, the presence of backflow is directly signaled by the density dropping below a critical threshold.

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